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Def:- Discontinuity :- A point at which a function is not continuous is called a point of discontinuity.

Types of Discontinuity :-

1. Removable discontinuity :-

If $\lim_{x \to a} f(x)$ Exist, and $\lim_{x \to a} f(x) \neq f(a)$ (and) $f(a)$ is not defined then we say that f has removable discontinuity at 'a'

Ex:- Consider $f(x) = \frac{x^2-4}{x-2} \quad \forall x \neq 2$ and $f(x) = 0$ for $x = 2$

we write $f(x) = \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$\therefore \lim_{x \to 2} (x+2) = 4$.

But given $f(2) = 0$

$\therefore \lim_{x \to 2} f(x) \neq f(2)$

Hence f has removable discontinuity at $x = 2$.

2. Jump discontinuity :-

If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exists and are not equal then we say that f has Jump discontinuity at 'a'.

ex:- Consider the function $f(x) = \frac{|x|}{x} \quad \forall x \neq 0$.

Now $|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x \geq 0 \end{cases}$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

$\therefore \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence f has Jump discontinuity at '0'

3) First Kind discontinuity :-

A removable discontinuity (or) Jump discontinuity of a function. is called simple discontinuity

4) Second Kind discontinuity :-

If atleast one of the limits of $\lim_{x\to a^-} f(x)$. (or) $\lim_{x\to a^+} f(x)$ not exists then f is said to have the discontinuity of second kind at $x=a$

Ex:- Consider the function $f(x) = \frac{1}{x-a}$ $\forall x \neq a$

and $f(a)=k$,

$\therefore \lim_{x\to a^-} f(x) = f(a-0) = -\infty$, $\lim_{x\to a^+} f(x) = f(a+0) = +\infty$

1. prob:- obtain the points of discontinuity of the function f. Defined by $f(0)=0$, $f(x) = \frac{1}{2}-x$ for $0<x<\frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2}$. $f(x) = (\frac{3}{2}-x)$ for $\frac{1}{2}<x<1$ and

$f(1)=1$ Examine the types of discontinuity:

Sol:- i) Continuity at '0':-

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{1}{2}-x\right) = \frac{1}{2} \neq f(0) \text{ as } f(0)=0$$

$\therefore$ f has removable discontinuity at $x=0$.

ii) Continuity at $\frac{1}{2}$:-

$\left[\text{ie, } \lim_{x\to\frac{1}{2}}\left[\frac{1}{2}-\frac{1}{2}\right]=0\right]$

$$\lim_{x\to\frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}}\left(\frac{1}{2}-x\right)=0$$

$$\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}}\left(\frac{3}{2}-x\right)=1$$

$\left[\text{ie, } \lim_{x\to\frac{1}{2}}\left[\frac{3}{2}-\frac{1}{2}\right] = 1\right]$

$$\therefore \lim_{x\to\frac{1}{2}^-} f(x) \neq \lim_{x\to\frac{1}{2}^+} f(x)$$

Hence f has jump discontinuity at $x=1/2$.

iii) Continuity at 1:-

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(\frac{3}{2}-x\right) = \frac{1}{2}$$

$$= \left(\frac{3}{2}-1\right) = \frac{1}{2} \neq f(1).$$

Hence f has removable discontinuity at $x=1$.

2) Prob:- Prove that the function defined by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$ is continuous at '0'

Sol:- Given $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Let $\varepsilon > 0$

consider $|f(x) - f(0)| < \varepsilon$

$\Rightarrow |x^2 \sin\left(\frac{1}{x}\right) - 0| < \varepsilon$

$\Rightarrow |x^2 \sin\frac{1}{x}| < \varepsilon$

$\Rightarrow |f(x) - f(0)| \leq |x^2| < \varepsilon \quad (\because |\sin x| \leq 1)$

$\Rightarrow |x| < \sqrt{\varepsilon}$

Now $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \sqrt{\varepsilon}$

ie, $|x - 0| < \sqrt{\varepsilon}$

Choose $\delta = \sqrt{\varepsilon}$

then $|x - 0| < \sqrt{\varepsilon} \Rightarrow |f(x) - f(0)| < \varepsilon$

$\therefore f$ is continuous at 0.

3) Prove that the function defined by $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$ is continuous at '0'.

Sol:- Given $f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Let $\varepsilon > 0$,

Consider $|f(x) - f(0)| < \varepsilon$.

$\Rightarrow |x \cdot \sin(\frac{1}{x}) - 0| < \varepsilon$

$\Rightarrow |x \cdot \sin \frac{1}{x}| < \varepsilon$

$\Rightarrow |f(x) - f(0)| \leq |x| < \varepsilon$ [since $|\sin x| \leq 1$]

$\Rightarrow |x| < \varepsilon$

Now $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \varepsilon$.

ie $|x - 0| < \varepsilon$ choose $\delta = \varepsilon$.

Then $|x - 0| < \varepsilon \Rightarrow |f(x) - f(0)| < \varepsilon$.

$\therefore$ f is continuous at '0'

4) Discuss the continuity of the function, $f(x) = x - [x]$. where $[x]$ is integral-part of x at the point $x = 3$.

Sol:- $f(x) = x - [x]$.

Now $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x - [x]$

$= \lim_{x \to 3^-} 3^- - [3^-]$

$= 3^- - 2$

$= 1^-$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x - [x]$

$= \lim_{x \to 3^+} 3^+ - [3^+]$

$= 3^+ - 3 = 0^+$

$\therefore \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

Hence $f(x)$ is not continuous at 3.

5) Let $f : R \to R$ be function, such that $f(x) = \frac{\sin(a+1)x + \sin x}{x}$

a) for $x<0$ and $f(x) = c$ for $x=0$ and

b) $f(x) = \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$ for $x>0$, Determine the value of a, b, c for which function is continuous at $x=0$

Sol:- Given $f(x) = \frac{\sin(a+1)x + \sin x}{x}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$= \lim_{x\to 0} \frac{\left[2\sin\left(\frac{(a+1)x + x}{2}\right) \cdot \cos\left(\frac{(a+1)x - x}{2}\right)\right]}{x}$$

[Hint:-

$\sin C + \sin D$

$= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$]

$$= \lim_{x\to 0} \frac{\left[2\sin\left(\frac{ax+x+x}{2}\right) \cdot \cos\left(\frac{ax+\cancel{x}-\cancel{x}}{2}\right)\right]}{x}$$

$$= \lim_{x\to 0} \left[\frac{2\sin\left(\frac{ax+2x}{2}\right) \cdot \cos\left(\frac{ax}{2}\right)}{x}\right]$$

$$= \lim_{x\to 0} \left[\frac{2\sin\left(\frac{a+2}{2}\right)x}{x}\right] \lim_{x\to 0} \cos\left(\frac{ax}{2}\right)$$

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} 2\sin\left(\frac{a+2}{\cancel{2}}\right) \cdot 1.$$

(since $\cos 0° = 1$)

$$= a+2$$

(since sin function లో x లేదు కాబట్టి Constant value)

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$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} = \left(\frac{x^{1/2} + (bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}\right) = \left(\frac{(x^{1/2} + b^{1/2}\cdot x^{1/2}\cdot x^{1/2}) - x^{1/2}}{bx^{3/2}}\right)$$

$$= \lim_{x\to 0^+} \frac{x^{1/2}\left[(1+bx)^{1/2} - 1\right]}{bx^{3/2}}$$

$$= \lim_{x\to 0^+} \frac{\cancel{x^{1/2}}\left[1+bx\right]^{1/2} - 1}{b\cancel{x^{1/2}}\cdot x^{1}}$$

$$= \lim_{x\to 0^+} \frac{(1+bx)^{1/2} - 1}{bx}$$

$$= \cancel{\lim_{x\to 0^+} \frac{1+(bx)^{1/2} - 1}{(bx)^{1/2}(bx)^{1/2}}}$$

$$= \text{So, } \lim_{x\to 0^+} \frac{\sqrt{1+bx} - 1}{bx} \times \frac{\sqrt{1+bx} + 1}{\sqrt{1+bx} + 1}$$

$$= \lim_{x\to 0^+} \frac{\cancel{1} + bx - \cancel{1}}{bx\left((1+bx)^{1/2} + 1\right)}$$

$$= \lim_{x\to 0^+} \frac{\cancel{bx}}{\cancel{bx}\,(1+bx)^{1/2} + 1}$$

$$= \lim_{x\to 0^+} \frac{1}{(1+bx)^{1/2} + 1}$$

$$= \frac{1}{2} \quad (\because b \neq 0)$$

This is independent of b. So, b may have any real value other than 0 since f is continuous at $x=0$ we have $a+2=c$

$$= 1/2$$

$\Rightarrow a+2 = \frac{1}{2}$

$\Rightarrow a = \frac{1}{2} - 2$

$\Rightarrow a = -\frac{3}{2}$

$\therefore a = -\frac{3}{2}$

$b \neq 0$ (is independent value)

$c = \frac{1}{2}$

6) Let $f: R \to R$ be a function such that $f(x) = \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}$ if $x \neq 0$. and $f(0) = 0$ Discuss the continuity at $x = 0$.

Sol:- Given $f(x) = \begin{cases} \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0 \end{cases}$

Now, $f(x) = \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}$

$= \frac{x \cdot \cancel{e^{1/x}}\left(1 - \frac{e^{-1/x}}{e^{1/x}}\right)}{\cancel{e^{1/x}}\left(1 + \frac{e^{-1/x}}{e^{1/x}}\right)}$

$= x \cdot \left(\frac{1 - e^{-2/x}}{1 + e^{-2/x}}\right)$

Now, $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} f(-h)$

$$= \lim_{h\to 0} (-h)\left[\frac{1-e^{+2/h}}{1+e^{2/h}}\right]$$

$$= 0$$

again $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} f(h).$$

$$= \lim_{h\to 0} h\frac{\left[1-e^{-2/h}\right]}{\left[1+e^{-2/h}\right]}$$

$$= 0.$$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

Hencee f is continuous at $x=0$